

Variational inequalities

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Abstract

If $-\infty < \alpha < \beta < \infty$ and $f \in C^3([\alpha, \beta] \times \mathbf{R}^2, \mathbf{R})$ is bounded, while $y \in C^2([\alpha, \beta], \mathbf{R})$ solves the typical one-dimensional problem of the calculus of variations to minimize the function

$$F(y) = \int_{\alpha}^{\beta} f(x, y(x), y'(x)) dx,$$

then for any $\phi \in C^2([\alpha, \beta], \mathbf{R})$ for which $\phi^{(k)}(\alpha) = \phi^{(k)}(\beta) = 0$ for every $k \in \{0, 1, 2\}$, we prove that $\int_{\alpha}^{\beta} \left(\frac{\partial^2 f}{\partial y^2} \phi^2 - \frac{\partial^3 f}{\partial y^2 \partial y'} 2\phi^3 \right) dx \geq \int_{\alpha}^{\beta} \left(\frac{\partial^2 f}{\partial y \partial y'} 2\phi \phi' + \frac{\partial^3 f}{\partial y \partial y'^2} 2\phi^2 \phi' + \frac{\partial^2 f}{\partial y'^2} \phi \phi'' + \frac{\partial^3 f}{\partial y \partial y'^2} \phi' \phi^2 + \frac{\partial^3 f}{\partial y'^3} \phi \phi'^2 \right) dx$, so either the above are variational inequalities of motion or the Lagrangian of motion is not C^3 .

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1. Definition. If $-\infty < \alpha < \beta < \infty$ and $f : [\alpha, \beta] \times \mathbf{R}^2 \rightarrow \mathbf{R}$ is any bounded continuous function, then the typical one-dimensional problem of the calculus of variations is to minimize the function F , which is defined by the relation

$$F(y) = \int_{\alpha}^{\beta} f(x, y(x), y'(x)) dx, \quad (1)$$

where $y : [\alpha, \beta] \rightarrow \mathbf{R}$ ranges over a suitably chosen class of functions.

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The following two propositions are well-known. See Appendix D on pages 151-152 of [1].

2. Proposition. If $-\infty < \alpha < \beta < \infty$ and r is any positive integer, while $y : [\alpha, \beta] \rightarrow \mathbf{R}$ is any continuous function such that

$$\int_{\alpha}^{\beta} y(x)\eta(x)dx = 0 \quad (2)$$

for every $\eta \in C^r([\alpha, \beta], \mathbf{R})$ for which

$$\eta^{(k)}(\alpha) = \eta^{(k)}(\beta) = 0 \quad (3)$$

for every $k \in \{0, 1, \dots, r\}$, then

$$y = 0 \quad (4)$$

on $[\alpha, \beta]$.

3. Proposition. If $-\infty < \alpha < \beta < \infty$ and $y : [\alpha, \beta] \rightarrow \mathbf{R}$ is any continuously differentiable function that solves the typical one-dimensional problem of the calculus of variations to minimize the function

$$F(y) = \int_{\alpha}^{\beta} f(x, y(x), y'(x)) dx, \quad (5)$$

where $f : [\alpha, \beta] \times \mathbf{R}^2 \rightarrow \mathbf{R}$ is C^2 , then

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) = 0. \quad (6)$$

4. Definition. Keeping the notation and the assumptions as in the previous proposition, given any $\phi \in C^2([\alpha, \beta], \mathbf{R})$ for which

$$\phi^{(k)}(\alpha) = \phi^{(k)}(\beta) = 0 \quad (7)$$

for every $k \in \{0, 1, 2\}$, it is not difficult to see that the function

$$I : \mathbf{R} \ni t \mapsto F(y + t\phi) \in \mathbf{R} \quad (8)$$

is twice differentiable and attains its minimum at $t = 0$, so, by virtue of Proposition 2 of Section 17.1.3 on page 409 of [2], as in Section 10.7 on page 458 of [3] or in Section 1.3 on page 16 of [4], one obtains that

$$\begin{aligned}
I''(t) &= \int_{\alpha}^{\beta} \left(\frac{\partial^2 f}{\partial y^2} (x, y(x) + t\phi(x), y'(x) + t\phi'(x)) \phi(x)^2 \right. \\
&+ \frac{\partial^2 f}{\partial y \partial y'} (x, y(x) + t\phi(x), y'(x) + t\phi'(x)) 2\phi(x)\phi'(x) \\
&+ \left. \frac{\partial^2 f}{\partial y'^2} (x, y(x) + t\phi(x), y'(x) + t\phi'(x)) \phi'(x)^2 \right) dx. \tag{9}
\end{aligned}$$

Our first purpose in this article is to prove the following.

5. Proposition. Keeping the notation as in the previous definition, if f is C^3 and y is C^2 , then

$$\begin{aligned}
I''(0) &= \int_{\alpha}^{\beta} \left(\frac{\partial^2 f}{\partial y^2} \phi^2 - \frac{\partial^3 f}{\partial y^2 \partial y'} 2\phi^3 \right) dx - \int_{\alpha}^{\beta} \left(\frac{\partial^2 f}{\partial y \partial y'} 2\phi\phi' \right. \\
&+ \left. \frac{\partial^3 f}{\partial y \partial y'^2} 2\phi^2\phi' + \frac{\partial^2 f}{\partial y'^2} \phi\phi'' + \frac{\partial^3 f}{\partial y \partial y'^2} \phi'\phi^2 + \frac{\partial^3 f}{\partial y'^3} \phi\phi'^2 \right) dx. \tag{10}
\end{aligned}$$

Proof. By virtue of Proposition 2 of Section 17.1.3 on page 409 of [2], one obtains that

$$I''(0) = \int_{\alpha}^{\beta} \left(\frac{\partial^2 f}{\partial y^2} \phi^2 + \frac{\partial^2 f}{\partial y \partial y'} 2\phi\phi' + \frac{\partial^2 f}{\partial y'^2} \phi'^2 \right) dx \tag{11}$$

$$= \int_{\alpha}^{\beta} \frac{\partial^2 f}{\partial y^2} \phi^2 + \int_{\alpha}^{\beta} \left(\frac{\partial^2 f}{\partial y \partial y'} 2\phi + \frac{\partial^2 f}{\partial y'^2} \phi' \right) \phi' dx \tag{12}$$

$$= \int_{\alpha}^{\beta} \frac{\partial^2 f}{\partial y^2} \phi^2 + \left[\left(\frac{\partial^2 f}{\partial y \partial y'} 2\phi + \frac{\partial^2 f}{\partial y'^2} \phi' \right) \phi \right]_{\alpha}^{\beta} - \int_{\alpha}^{\beta} \left(\frac{\partial^2 f}{\partial y \partial y'} 2\phi + \frac{\partial^2 f}{\partial y'^2} \phi' \right)' \phi dx \tag{13}$$

$$= \int_{\alpha}^{\beta} \frac{\partial^2 f}{\partial y^2} \phi^2 - \int_{\alpha}^{\beta} \left(\frac{\partial^2 f}{\partial y \partial y'} 2\phi + \frac{\partial^2 f}{\partial y'^2} \phi' \right)' \phi dx \tag{14}$$

$$\begin{aligned}
&= \int_{\alpha}^{\beta} \frac{\partial^2 f}{\partial y^2} \phi^2 - \int_{\alpha}^{\beta} \left(\frac{\partial^2 f}{\partial y \partial y'} 2\phi' + \left(\frac{\partial^3 f}{\partial x \partial y \partial y'} \cdot 0 + \frac{\partial^3 f}{\partial y^2 \partial y'} \phi + \frac{\partial^3 f}{\partial y \partial y'^2} \phi' \right) 2\phi \right. \\
&+ \left. \frac{\partial^2 f}{\partial y'^2} \phi'' + \left(\frac{\partial^3 f}{\partial x \partial y'^2} \cdot 0 + \frac{\partial^3 f}{\partial y \partial y'^2} \phi + \frac{\partial^3 f}{\partial y'^3} \phi' \right) \phi' \right) \phi dx \tag{15}
\end{aligned}$$

$$\begin{aligned}
&= \int_{\alpha}^{\beta} \frac{\partial^2 f}{\partial y^2} \phi^2 - \int_{\alpha}^{\beta} \left(\frac{\partial^2 f}{\partial y \partial y'} 2\phi' + \left(\frac{\partial^3 f}{\partial y^2 \partial y'} \phi + \frac{\partial^3 f}{\partial y \partial y'^2} \phi' \right) 2\phi \right. \\
&+ \left. \frac{\partial^2 f}{\partial y'^2} \phi'' + \left(\frac{\partial^3 f}{\partial y \partial y'^2} \phi + \frac{\partial^3 f}{\partial y'^3} \phi' \right) \phi' \right) \phi dx \tag{16}
\end{aligned}$$

$$\begin{aligned}
&= \int_{\alpha}^{\beta} \left(\frac{\partial^2 f}{\partial y^2} \phi^2 - \frac{\partial^3 f}{\partial y^2 \partial y'} 2\phi^3 \right) dx - \int_{\alpha}^{\beta} \left(\frac{\partial^2 f}{\partial y \partial y'} 2\phi\phi' \right. \\
&+ \left. \frac{\partial^3 f}{\partial y \partial y'^2} 2\phi^2\phi' + \frac{\partial^2 f}{\partial y'^2} \phi\phi'' + \frac{\partial^3 f}{\partial y \partial y'^2} \phi'\phi^2 + \frac{\partial^3 f}{\partial y'^3} \phi\phi'^2 \right) dx. \tag{17}
\end{aligned}$$

Our second purpose in this article is to prove the following.

6. Proposition. Keeping the notation as in the previous proposition, for any such ϕ , we have that

$$\begin{aligned} \int_{\alpha}^{\beta} \left(\frac{\partial^2 f}{\partial y^2} \phi^2 - \frac{\partial^3 f}{\partial y^2 \partial y'} 2\phi^3 \right) dx &\geq \int_{\alpha}^{\beta} \left(\frac{\partial^2 f}{\partial y \partial y'} 2\phi \phi' \right. \\ &\quad \left. + \frac{\partial^3 f}{\partial y \partial y'^2} 2\phi^2 \phi' + \frac{\partial^2 f}{\partial y'^2} \phi \phi'' + \frac{\partial^3 f}{\partial y \partial y'^2} \phi' \phi^2 + \frac{\partial^3 f}{\partial y'^3} \phi \phi'^2 \right) dx, \end{aligned} \quad (18)$$

so either the above are variational inequalities of motion or the Lagrangian of motion is not C^3 .

Proof. It is enough to notice that

$$I''(0) \geq 0. \quad (19)$$

7. Remark. Keeping the notation as in the previous proposition, one may take

$$\phi(x) = \lambda ((x - \alpha)(x - \beta))^n, \quad (20)$$

where $x \in [\alpha, \beta]$, while $\lambda > 0$ and $n \in \mathbb{N} \setminus \{0, 1, 2\}$.

8. Example. If we consider the simple pendulum, where g is the acceleration of gravity and ℓ is the length of the weightless thread to the one end of which is connected a particle of mass m , while θ is the angular displacement as a function of time t , then the Lagrangian of the simple pendulum is

$$L = \frac{1}{2} m \ell^2 \dot{\theta}^2 + mg\ell \cos \theta \quad (21)$$

and it is C^∞ , so

$$\frac{\partial L}{\partial t} = 0, \quad (22)$$

$$\frac{\partial L}{\partial \theta} = -mg\ell \sin \theta \quad (23)$$

and

$$\frac{\partial L}{\partial \dot{\theta}} = m\ell^2 \dot{\theta}, \quad (24)$$

which imply that the Euler-Lagrange equation

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) = 0 \quad (25)$$

in Proposition 3 takes the form

$$\ddot{\theta} + \frac{g}{\ell} \sin \theta = 0. \quad (26)$$

So, apart from solving equation (26), θ must satisfy the conclusion of Proposition 6 and Remark 7. Since

$$\frac{\partial^2 L}{\partial \theta^2} = -mg\ell \cos \theta, \quad (27)$$

$$\frac{\partial^2 L}{\partial \theta \partial \dot{\theta}} = 0 \quad (28)$$

and

$$\frac{\partial^2 L}{\partial \dot{\theta}^2} = m\ell^2, \quad (29)$$

while

$$\frac{\partial^3 L}{\partial \theta^2 \partial \dot{\theta}} = 0, \quad (30)$$

$$\frac{\partial^3 L}{\partial \theta \partial \dot{\theta}^2} = 0 \quad (31)$$

and

$$\frac{\partial^3 L}{\partial \dot{\theta}^3} = 0, \quad (32)$$

it follows that if

$$\phi(t) = \lambda ((t - \alpha)(t - \beta))^n, \quad (33)$$

where $t \in [\alpha, \beta]$, while $\lambda > 0$ and $n \in \mathbf{N} \setminus \{0, 1, 2\}$, then

$$\begin{aligned} & \int_{\alpha}^{\beta} (-mg\ell \cos \theta \cdot \phi^2 - 0 \cdot 2\phi^3) dt \\ & \geq \int_{\alpha}^{\beta} (0 \cdot 2\phi\dot{\phi} + 0 \cdot 2\phi^2\dot{\phi} + m\ell^2 \cdot \phi\ddot{\phi} + 0 \cdot \dot{\phi}\phi^2 + 0 \cdot \phi\dot{\phi}^2) dt \end{aligned} \quad (34)$$

or equivalently

$$-mg\ell \int_{\alpha}^{\beta} \cos \theta(t) \phi(t)^2 dt \geq m\ell^2 \int_{\alpha}^{\beta} \phi(t) \ddot{\phi}(t) dt \quad (35)$$

or equivalently

$$-g \int_{\alpha}^{\beta} \cos \theta(t) \phi(t)^2 dt \geq \ell \int_{\alpha}^{\beta} \phi(t) \ddot{\phi}(t) dt \quad (36)$$

or equivalently

$$-g \int_{\alpha}^{\beta} \cos \theta(t) \phi(t)^2 dt \geq \ell \left([\phi(t) \dot{\phi}(t)]_{t=\alpha}^{t=\beta} - \int_{\alpha}^{\beta} \dot{\phi}(t) \dot{\phi}(t) dt \right) \quad (37)$$

or equivalently

$$g \int_{\alpha}^{\beta} \cos \theta(t) \phi(t)^2 dt \leq \ell \int_{\alpha}^{\beta} \dot{\phi}(t)^2 dt. \quad (38)$$

A formula for θ , via (26), can be derived from Section 2.1 on pages 69-80 of [5], so if

$$0 < \theta_0 < \pi \quad (39)$$

and

$$\dot{\theta}_0 = 0, \quad (40)$$

then

$$t = \sqrt{\frac{\ell}{g}} \ln \left(\frac{\tan \left(\frac{\pi}{4} - \frac{\theta_0}{4} \right)}{\tan \left(\frac{\pi}{4} - \frac{\theta}{4} \right)} \right) \quad (41)$$

and consequently

$$\theta(t) = \pi - 4 \arctan \left(e^{-t\sqrt{\frac{g}{\ell}}} \tan \left(\frac{\pi}{4} - \frac{\theta_0}{4} \right) \right) \quad (42)$$

must satisfy (38) for all ϕ in question.

References

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